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GENERAL BOUNDS ON EDGE-CHROMATIC SUMS OF GRAPHS

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ABSTRACT

Proper edge-coloring of a graph is a mapping from its edges to natural numbers so that adjacent edges receive different numbers (colors). The edge-chromatic sum of the graph is the minimum sum of the colors on all edges among all proper edge-colorings of the graph. This work aims to find lower and upper bounds of the edge-chromatic sum parameter of graphs. Two lower and two upper bounds are obtained, they are compared with some existing results. It is also shown that both bounds are efficient for certain classes of graphs and for each bound there are graphs, for which it is more efficient than the other one.

Keywords: edge-chromatic sum, proper edge-coloring.

Introduction

A proper vertex-coloring (edge-coloring) of a graph G is a mapping from its vertices (edges) to numbers $1, 2, 3, \dots$ so that adjacent vertices (edges) map to different numbers. The vertex-chromatic (edge-chromatic) sum of the graph is the minimum sum of colors on all vertices (edges) in any proper vertex-coloring (edge-coloring). It is denoted by $\Sigma(G)$ ($\Sigma'(G)$).

The problem of finding the vertex-chromatic sum of the graph or shortly the chromatic sum problem was first suggested by Supowit in 1987 [1] and independently by Kubicka and Schwenk in 1989 [2]. Similarly, in 1998, Bar-Noy et al defined the problem of finding the edge chromatic sum [3]. The problems have applications in integrated circuit design and resource allocation [1, 2, 3]. Both problems are NP-complete in general [2, 3] but a lot of estimation results are known for both parameters. Thomassen et al. [4] showed that for any graph G , $\Sigma(G) \leq |V(G)| + |E(G)|$. They also showed that for any graph G with no isolated vertices $\lceil \sqrt{8|E(G)|} \rceil \leq \Sigma(G) \leq 3|E(G)|$. Kokosiński and Kwarciński obtained $\Sigma(G) \geq |V(G)| + \frac{s(G)(s(G)-1)}{2}$ for any graph in [5]. Other general bounds for $\Sigma(G)$ can be found in [5, 6, 7]. Salavatipour found the exact value of the edge-chromatic sum of chain bipartite graphs and a tight upper bound for split graphs [8]. Petrosyan and Kamalian provided the exact value of the edge-chromatic sum of complete graphs

and two upper bounds for certain split graphs [9]. Petrosyan found an upper bound on edge-chromatic sum of almost regular graphs in [10].

Despite many results on specific graph types, a comprehensive analysis of the lower and upper bounds of edge-chromatic sums for graphs in general is an area of examination yet. Our goal is to find the lower and upper bounds of edge-chromatic sums depending on the main parameters of the graph, such as vertex count, edge count, the degrees of the graph vertices, etc. We compare the existing and newly suggested bounds, provide graph types for which the bounds are tight, and provide graph types for each bound for which they are more efficient than the others.

Definitions

We consider only finite, undirected, and simple graphs. Basic terms not defined here (graph, vertex, edge, etc.) can be found in [11]. We denote the vertex and edge sets of graph G by $V(G)$ and $E(G)$ respectively. We denote by $d_G(v)$ the degree of the vertex v in the graph G and $\Delta(G)$ is the maximum degree in graph G . Line graph $L(G)$ of graph G is the graph with vertex set $E(G)$ and the two vertices of $L(G)$ are connected by an edge if and only if the corresponding edges have a common vertex in G .

The chromatic index of graph G is denoted by $\chi'(G)$ and the maximum matching size is denoted by $\alpha'(G)$. Proper edge-colorings, for which the sum of colors on all edges is equal to $\Sigma'(G)$ are called sum edge-colorings and the minimal number of colors needed for a sum edge-colorings is called the edge-strength of the graph G and is denoted by $s'(G)$. Edge-colorings that use c colors will be written edge c -colorings sometimes. Obviously, $\Delta(G) \leq \chi'(G) \leq s'(G)$. Note that Hajiabolhasan [12] proved that $s'(G) \leq \Delta(G) + 1$.

Main Results

Lower Bounds

Consider a graph G . First, consider the lower bounds on $\Sigma'(G)$.

This simple result is mentioned as a fact that immediately derives from the definition of edge-coloring in [13] and is also provided as a result in the context of line graphs in [3].

Lemma 1. For any graph G , we have

$$\Sigma'(G) \geq \left\lceil \frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{1}{2} |E(G)| \right\rceil.$$

Proof. Consider the sum edge-coloring α of graph G . For any $v \in V(G)$,

$$\sum_{u \in N_G(v)} \alpha(vu) \geq 1 + 2 + \dots + d_G(v) = \frac{d_G(v)(d_G(v) + 1)}{2},$$

since the colors on the edges adjacent to v are natural numbers that are pairwise distinct. On the other hand,

$$\begin{aligned}
 \sum' (G) &= \sum_{e \in E(G)} \alpha(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{u \in N_G(v)} \alpha(vu) \\
 &\geq \frac{1}{2} \sum_{v \in V(G)} \frac{d_G(v)(d_G(v) + 1)}{2} \\
 &= \frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{1}{4} \sum_{v \in V(G)} d_G(v) \\
 &= \frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{1}{2} |E(G)|.
 \end{aligned}$$

If we use the Sedrakyan's inequality as follows:

$$\sum_{v \in V(G)} \frac{d_G^2(v)}{1} \geq \frac{(\sum_{v \in V(G)} d_G(v))^2}{|V(G)|} = \frac{4|E(G)|^2}{|V(G)|},$$

we can obtain a lower bound that depends only on the number of vertices and edges:

Corollary 1. For any graph G ,

$$\sum' (G) \geq \left\lfloor \frac{|E(G)|^2}{|V(G)|} + \frac{1}{2} |E(G)| \right\rfloor.$$

Since $\Sigma'(G) = \Sigma(L(G))$, existing results on the lower bound of $\Sigma(G)$ can be translated in edge-coloring terms. For instance, note that Bar-Noy et al. [3] provided a lower bound $\Sigma(G) \geq |V(G)| + \frac{|E(G)|}{\Delta(G)}$ in the proof of their Theorem 4.3. If we use $\Sigma'(G) = \Sigma(L(G))$ to find a similar result for edge-chromatic sums as follows:

$$\begin{aligned}
 \Sigma'(G) &= \Sigma(L(G)) \geq |V(L(G))| + \frac{|E(L(G))|}{\Delta(L(G))} \\
 &= |E(G)| + \frac{\sum_{v \in V(G)} d_G(v)(d_G(v) - 1)}{2\Delta(L(G))},
 \end{aligned}$$

then considering $\Delta(L(G)) \leq 2\Delta(G) - 2$, this will not give us a better result than Lemma 1 gives us (considering $\Delta(G) > 1$):

$$\begin{aligned}
 \frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{|E(G)|}{2} - \left(|E(G)| + \frac{\sum_{v \in V(G)} d_G(v)(d_G(v) - 1)}{4\Delta(G) - 4} \right) &\geq \\
 &\geq \frac{(\Delta(G) - 2)(\sum_{v \in V(G)} d_G^2(v) - 2|E(G)|)}{4\Delta(G) - 4} \geq 0.
 \end{aligned}$$

There is also a lower bound of $\Sigma(G)$ obtained by Thomassen et al in [4], which gives us that

$$\begin{aligned} \sum' (G) &= \sum (L(G)) \geq \left\lceil \sqrt{8|E(L(G))|} \right\rceil \\ &= \left\lceil 2 \sqrt{\sum_{v \in V(G)} d_G(v)(d_G(v) - 1)} \right\rceil. \end{aligned}$$

But if we compare the squares of numbers $\frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{1}{2}|E(G)|$ and $2 \sqrt{\sum_{v \in V(G)} d_G(v)(d_G(v) - 1)}$, we will see that the first expression is always greater when $|E(G)| > 4$ (let us denote $\sum_{v \in V(G)} d_G^2(v)$ by x for simplicity):

$$\begin{aligned} \left(\frac{x + 2|E(G)|}{4} \right)^2 - 4 \sum_{v \in V(G)} d_G(v)(d_G(v) - 1) &= \\ = \frac{(x + 2|E(G)|)^2 - 64x + 128|E(G)|}{16} &= \\ = \frac{(x + 2|E(G)| - 32)^2 + 256(|E(G)| - 4)}{16} \geq 0. \end{aligned}$$

By checking all the remaining graphs where $|E(G)| \leq 4$ manually, we obtain that the lower bound given in Lemma 1 is always not less optimal.

Now we obtain one more lower bound.

Lemma 2.

For a graph G and any integer k that satisfies $0 \leq k \leq s'(G) + 1$, we have:

$$\begin{aligned} \sum' (G) &\geq k \left(|E(G)| - \frac{\alpha'(G)(k-1)}{2} \right) + \\ &\quad + \frac{(s'(G) - k)(s'(G) - k + 1)}{2}. \end{aligned}$$

Proof. Let α be any sum edge $s'(G)$ -coloring of G . For each color i ($1 \leq i \leq s'(G)$), denote the number of edges that are assigned with the color i in α by c_i . We know that $1 \leq c_i \leq \alpha'(G)$. To complete the proof, we can see that for any $0 \leq k \leq s'(G)$,

$$\begin{aligned} \sum' (G) &= \sum_{i=1}^{s'(G)} i \cdot c_i = \sum_{i=1}^{s'(G)} (k + (i - k)) \cdot c_i = \\ &= \sum_{i=1}^{s'(G)} k \cdot c_i - \sum_{i=1}^k (k - i) \cdot c_i + \\ &+ \sum_{i=k+1}^{s'(G)} (i - k) \cdot c_i = k|E(G)| - \sum_{i=1}^k (k - i) \cdot c_i + \sum_{i=k+1}^{s'(G)} (i - k) \cdot c_i \geq \end{aligned}$$

$$\begin{aligned}
 &\geq k|E(G)| - \sum_{i=1}^k (k-i) \cdot \alpha'(G) + \sum_{i=k+1}^{s'(G)} (i-k) \cdot 1 = \\
 &= k \left(|E(G)| - \frac{\alpha'(G)(k-1)}{2} \right) \\
 &\quad + \frac{(s'(G)-k)(s'(G)-k+1)}{2}.
 \end{aligned}$$

Note that under $\sum_{i=a}^b$ we understand 0 if $a > b$. If we do the division of the summands not between k and $k+1$, but between $k-1$ and k , we can obtain the same result for $k = s'(G) + 1$ as well.

For the vertex-chromatic sum, Kokosiński and Kwarciány provided a lower bound, edge-coloring version of which can be obtained if we put $k = 1$ in the lemma above. Lecat, Lucet, and Li [7] obtained a result on $\Sigma(G)$, which, if translated into edge-coloring terms, is similar to the result of the lemma above. They claim that the best estimate is obtained when $k = \left\lfloor \frac{|E(G)| - s'(G)}{\alpha'(G) - 1} \right\rfloor + 1$.

Since $\alpha'(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and $s'(G) \geq \Delta(G)$, we can obtain the following corollary as well:

Corollary 2. For a graph G and any integer k that satisfies $0 \leq k \leq \Delta(G) + 1$, we have:

$$\sum'(G) \geq k \left(|E(G)| - \frac{k-1}{2} \left\lfloor \frac{|V(G)|}{2} \right\rfloor \right) + \frac{(\Delta(G)-k)(\Delta(G)-k+1)}{2}.$$

Note that here the upper bound of k is set to $\Delta(G) + 1$ instead of $s'(G) + 1$ since the inequality $(s'(G)-k)(s'(G)-k+1) \geq (\Delta(G)-k)(\Delta(G)-k+1)$ is not true for $\Delta(G) + 1 < k \leq s'(G) + 1$.

Upper Bounds

Now we proceed to the upper bounds.

Lemma 3. For any graph G ,

$$\sum'(G) \leq \frac{1}{2} \sum_{v \in V(G)} d_G^2(v).$$

Proof. We use the fact $\sum'(G) = \sum(L(G))$ here. Thomassen et al. also showed [4] that $\sum(G) \leq |V(G)| + |E(G)|$. By placing $L(G)$ instead of G , we get

$$\begin{aligned}
 \sum(L(G)) &\leq |V(L(G))| + |E(L(G))| \\
 &= |E(G)| + \sum_{v \in V(G)} \frac{d_G(v)(d_G(v)-1)}{2} = \\
 &= |E(G)| + \frac{1}{2} \sum_{v \in V(G)} d_G^2(v) - \frac{1}{2} \sum_{v \in V(G)} d_G(v) = \frac{1}{2} \sum_{v \in V(G)} d_G^2(v).
 \end{aligned}$$

There is an upper bound obtained by Thomassen that states that $\Sigma'(G) \leq 3|E(G)|$ if G has no isolated vertices. This means that if there are no adjacent vertices of degree 1 in G , then $\Sigma'(G) \leq \frac{3}{2} \sum_{v \in V(G)} d_G(v)(d_G(v) - 1)$. Let us compare this upper bound with the result of the lemma above. We see that $\frac{3}{2} \sum_{v \in V(G)} d_G(v)(d_G(v) - 1) - \frac{1}{2} \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} d_G(v)(d_G(v) - \frac{3}{2}) = \sum_{uv \in E(G)} (d_G(u) + d_G(v) - 3)$. Since for each edge $uv \in E(G)$, at least one from u and v has a degree greater than 1, this sum is always non-negative. This gives us that the upper bound of the lemma is more optimal.

We prove one more upper bound.

Lemma 4. For any graph G ,

$$\Sigma'(G) \leq \frac{1}{2}|E(G)|(\chi'(G) + 1).$$

Proof. Consider any proper edge-coloring of G that uses $\chi'(G)$ colors. Denote by c_i the number of edges colored with i . Obviously, we can reorder the colors so that $c_1 \geq c_2 \geq \dots \geq c_{\chi'(G)}$.

We use the Chebyshev's sum inequality, which says that if we have the numbers $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$, then

$$\frac{1}{n} \sum_{k=1}^n a_k b_k \leq \left(\frac{1}{n} \sum_{k=1}^n a_k \right) \left(\frac{1}{n} \sum_{k=1}^n b_k \right).$$

If we replace n with $\chi'(G)$, a_i ($i = 1, 2, \dots, n$) with c_i , and b_i with $1, 2, \dots, n$, we get

$$\frac{1}{\chi'(G)} \sum_{k=1}^{\chi'(G)} k c_k \leq \left(\frac{1}{\chi'(G)} \sum_{k=1}^{\chi'(G)} c_k \right) \left(\frac{1}{\chi'(G)} \sum_{k=1}^{\chi'(G)} k \right).$$

$\Sigma'(G) \leq \sum_{k=1}^{\chi'(G)} k c_k$, and the number $\sum_{k=1}^{\chi'(G)} c_k$ is exactly $|E(G)|$. Placing those values in the inequality above, we get

$$\Sigma'(G) \leq |E(G)| \frac{1}{\chi'(G)} \frac{\chi'(G)(\chi'(G) + 1)}{2} = \frac{|E(G)|(\chi'(G) + 1)}{2}.$$

Note that there is also a similar upper bound for vertex-chromatic sums, which is briefly described in [14]. Giaro and Kubale [13] mention an upper bound for bipartite graphs that is always less optimal than the result in the lemma above, since $\chi'(G) = \Delta(G)$ for bipartite graphs.

Since $\chi'(G) \leq \Delta(G) + 1$, we can write also the following:

Corollary 3. For any graph G ,

$$\Sigma'(G) \leq \frac{1}{2}|E(G)|(\Delta(G) + 2).$$

By collecting the results from the lemmas, we formulate a theorem about the lower and upper bounds of any graph:

Theorem 1.

For any graph G , we have

$$\sum'(G) \geq \left\lceil \frac{1}{4} \sum_{v \in V(G)} d_G^2(v) + \frac{1}{2} |E(G)| \right\rceil, \quad (1)$$

$$\begin{aligned} \sum'(G) \geq k \left(|E(G)| - \frac{\alpha'(G)(k-1)}{2} \right) \\ + \frac{(s'(G) - k)(s'(G) - k + 1)}{2} \end{aligned} \quad (2)$$

$(0 \leq k \leq s'(G) + 1),$

$$\sum'(G) \leq \frac{1}{2} \sum_{v \in V(G)} d_G^2(v), \quad (3)$$

$$\sum'(G) \leq \frac{1}{2} |E(G)| (\chi'(G) + 1). \quad (4)$$

In order to show the efficiency of each lower and upper bound, we provide:

1. Graphs, for which the bound is tight,
2. Graphs, for which the bound is better than the other bound.

First, consider r -regular graphs of order n , where $r \geq 2$. In this case, from (1), we obtain $\sum'(G) \geq \frac{nr(r+1)}{4}$, and (4) gives us that $\sum'(G) \leq \frac{nr(s'(G)+1)}{4}$. For class 1 graphs among such graphs (which are known to exist), we have $s'(G) = r$. Hence, the two bounds coincide, yielding a tight bound in both cases.

Now consider the star graphs. If we put $k = 0$ in the bound (2), we get $\sum'(G) \geq \frac{s'(G)(s'(G)+1)}{2}$, and the bound (4) gives us that $\sum'(G) \leq \frac{|E(G)|(\chi'(G)+1)}{2}$. It is not hard to see that the bounds again coincide, yielding tight bounds for (2) and (4).

Note that for star graphs the bound (1) gives $\sum'(G) \geq \left\lceil \frac{(|V(G)|-1)(|V(G)|+2)}{4} \right\rceil$, so for $|V(G)| > 3$, it becomes not more optimal than the bound (2).

To show that the second bound is not always more optimal than the first one, let us consider the graph provided in the Figure 1. (1) gives us the bound of 11 while (2) gives us the maximum bound 10 when $k = 2$.

From the upper bounds, (4) is more efficient than (3) in r -regular graphs with $r > 2$:

$$\frac{nr(\chi'(G) + 1)}{4} \leq \frac{nr(r + 2)}{4} = \frac{nr^2}{4} + \frac{nr}{2} < \frac{nr^2}{4} + \frac{nr}{2} = \frac{nr^2}{2}.$$

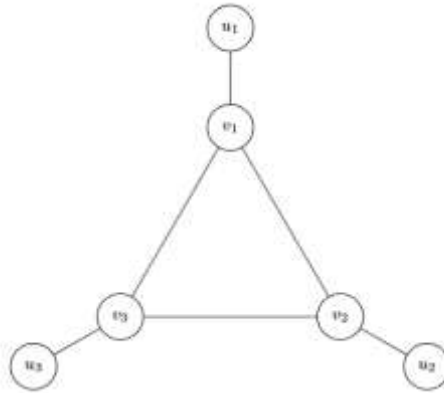


Figure 1: Graph with an edge-chromatic sum of 12

(3) is more efficient than (4) for example in wheel graphs (Figure 2): (3) gives the bound $\frac{n^2+9n}{2}$, while (4) gives $n(\chi'(W_{n+1}) + 1) \geq n(n+1)$. Obviously, for larger n , the upper bound (3) is more efficient.

Thereby, we showed, that for each bound there are graphs, where it is more optimal, and in some cases, they reach the exact value of the edge-chromatic sum.

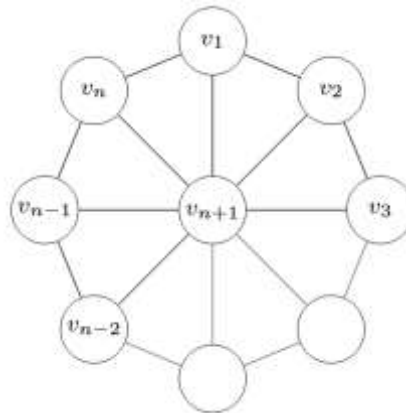


Figure 2: A wheel graph W_{n+1}

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ОБЩИЕ ОЦЕНКИ ДЛЯ РЕБЕРНО-ХРОМАТИЧЕСКИХ СУММ ГРАФОВ

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АННОТАЦИЯ

Правильная реберная раскраска графа – это отображение его ребер на натуральные числа таким образом, что смежные ребра получают разные числа (цвета). Реберно-хроматическая сумма графа – это минимальная сумма цветов на всех ребрах среди всевозможных правильных реберных раскрасок. Цель данной работы – найти нижние и верхние оценки для параметра реберно-хроматической суммы графов. Получены две нижние и две верхние оценки, которые сравнены с некоторыми существующими результатами. Также показано, что обе оценки эффективны для определенных классов графов, и для каждой границы существуют графы, для которых она более эффективна, чем другая.

Ключевые слова: реберно-хроматическая сумма, правильная реберная раскраска.