DOI 10.24412/1829-0450-fm-2025-1-26-32
УДК 519.17
Поступила: 19.05.2025г.
Сдана на рецензию: 19.05.2025г.
Подписана к печати: 21.05.2025г.

ON VERTEX DISTINGUISHING PROPER EDGE COLORINGS OF THE CARTESIAN SUM OF GRAPHS

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ABSTRACT

A proper edge coloring of a graph G is a mapping $f: E(G) \to Z_{\geq 0}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges $e, e' \in E(G)$. A proper edge coloring f of a graph G is called vertex distinguishing if for any different vertices $u, v \in V(G)$, $S(u, f) \neq S(v, f)$, where $S(v, f) = \{f(e) \mid e = vw \in E(G)\}$. The minimum number of colors required for a vertex distinguishing proper coloring of a graph G is denoted by $\chi'_{vd}(G)$ and called the vertex distinguishing chromatic index of G. In this work, we provide an upper bound on the vertex distinguishing chromatic index of the Cartesian sum of graphs.

Keywords: edge-coloring, vertex distinguishing edge-coloring, vertex distinguishing chromatic index, cartesian sum of graphs.

Introduction

All graphs discussed in this paper are finite, undirected, and contain neither loops nor multiple edges. For terminologies and notations not defined here, we primarily refer to West's book [1]. Let V(G) and E(G) denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex v $\in$ V(G) is denoted by $d_G(v)$ and the maximum degree of G by

$\Delta(G)$. A proper edge coloring of a graph G is a mapping $f: E(G) \to \mathbb{Z}_{\geq 0}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If f is a proper edge coloring of a graph G and $v \in V(G)$, then the *spectrum of a vertex* v, denoted by $S(v, f)$, is the set of all colors appearing on edges incident to v. We use the standard notations P_n, K_n and $K_{m,n}$ for the simple path, the complete graph on n vertices and the complete bipartite graph with m vertices in one part and n vertices in the other part of the bipartition, respectively.

The proper edge coloring f of a graph G is a vertex distinguishing proper coloring (abbreviated *VDP-coloring*) of G if $S(u, f) \neq S(v, f)$ for any two distinct vertices u and v in G. The minimum number of colors required for a *VDP-coloring* of a graph G without isolated edges and with at most one isolated vertex is called the vertex distinguishing chromatic index (abbreviated *VDP-chromatic index*) and denoted by $\chi'_{vd}(G)$. The concept of vertex distinguishing proper edge colorings of graphs was introduced by Burris and Schelp in [2] and, independently, as observability of a graph, by Cerný, Hornák and Soták [3]. In [2-6], the vertex distinguishing proper edge colorings of paths, cycles, complete, complete bipartite and multipartite graphs were investigated. In particular, the authors determined the vertex distinguishing chromatic index of some families of graphs. The following results have been proved by Burris and Schelp [2].

Theorem 1. If $n \geq 3$, then

$$\chi'_{vd}(K_n) = \begin{cases} n, if\ n\ is\ odd, \\ n+1, if\ n\ is\ even. \end{cases}$$

Theorem 2. Let m and n be any natural numbers. Then

$$\chi'_{vd}(K_{m,n}) = \begin{cases} n+1, \text{если}\ n > m \geq 2, \\ n+2, \text{если}\ n = m \geq 2. \end{cases}$$

The classical theorem by Konig [7] on proper edge coloring of bipartite graphs states
the following.

Theorem 3. For any bipartite graph G, we have

$$\chi'(G) = \Delta(G)$$

For any graphs G and H, let $G \circ H$ be the lexicographic product of G and H. The vertex set of $G \circ H$ is $V(G) \times V(H)$ and the edge set is defined below:

$$E(G \circ H) = \{(x,x')(y,y') \mid xy \in E(G) \text{ or } x = y \text{ and } x'y' \in E(H)\}$$

In [8], Baril, Kheddouci and Togni investigated vertex distinguishing proper edge colorings of Cartesian, direct, strong and lexicographic products of graphs. They derived upper bounds on the vertex distinguishing chromatic index of these products of graphs in terms of the vertex distinguishing chromatic indices of the factors. The following result has been proved by Baril, Kheddouci and Togni in [8].

Theorem 4. For any two connected graphs G and H different from K_2, we have

$$\chi'_{vd}(G \circ H) \leq \chi'_{vd}(G) + \chi'_{vd}(H) + (|V(H)| - 1)\,\chi'(G)$$

For any graphs G and H, the Cartesian sum of G and H, denoted by $G \oplus H$, has the vertex set $V(G) \times V(H)$ and edge set

$$E(G \oplus H) = \{(x,x')(y,y') \mid xy \in E(G) \text{ or } x'y' \in E(H)\}$$

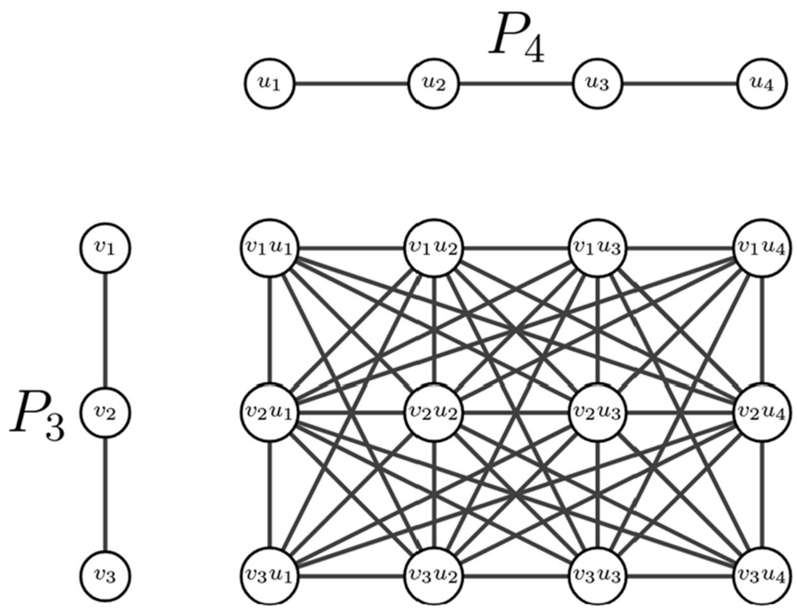

Figure 1. The Cartesian sum of P_3 and P_4.

This notion of graph product was introduced by Ore [9] in 1962.

In this work, we give an upper bound on the vertex distinguishing chromatic index of the Cartesian sum of graphs.

Main Result

Theorem 5. For any simple graphs G and H without isolated edges and with at most one isolated vertex, we have

$$\chi'_{vd}(G \oplus H) \leq \chi'_{vd}(G) + \chi'_{vd}(H) + (|V(H)| - 1)\,\chi'(G) + \\ + \chi'(\bar{G}) \cdot \Delta(H)$$

Proof. Let G and H be graphs with vertex sets $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_n\}$ respectively, such that they are not isomorphic to K_2. Note that for any vertex $v_i \in V(G)$ $(1 \leq i \leq n)$, the subgraph of the graph $G \oplus H$ induced by the vertices $(v_i, u_1), (v_{i,} u_2), \dots, (v_i, u_m)$ is isomorphic to graph H. We denote this subgraph by H_i $(1 \leq i \leq n)$. Since H_i is isomorphic to H, there exists a vertex distinguishing edge coloring f_{H_i} with colors $1,2, \dots, \chi'_{vd}(H)$ $(1 \leq i \leq n)$.

Let us now define the colors of edges between different subgraphs H_i and H_j of $G \oplus H$. The subgraphs H_i and H_j are called neighboring subgraphs if $v_i v_j \in E(G)$ $(1 \leq i \leq j \leq n)$. Let us denote by $W_{i,j}$ the subgraph of $G \oplus H$ formed by the vertex set $V(H_i) \cup V(H_j)$ and the edge set $\{uv : u \in V(H_i) \text{ and } v \in V(H_j)\}$.

When H_i and H_j are neighboring subgraphs, $W_{i,j}$ is isomorphic to the complete bipartite graph $K_{m,m}$. The edge set of $K_{m,m}$ can be partitioned into m perfect matchings. For each subgraph $W_{i,j}$, let us choose one of the m perfect matchings and apply the subgraph contraction operation. The resulting subgraph is isomorphic to graph G, hence there exists a vertex distinguishing edge coloring f_G with colors $\chi'_{vd}(H) + 1, \chi'_{vd}(H) + 2, \dots, \chi'_{vd}(H) + \chi'_{vd}(G)$ for the subgraph. For the edges of the perfect matching, we use the colors resulting from their contraction. For each of the remaining $m - 1$ perfect matchings, we also apply the contraction operation. The resulting subgraph is isomorphic to G, meaning that there exists a proper edge coloring with $\chi'(G)$ colors. Therefore, the edges of the remaining $m-1$ perfect matchings can be colored with colors $\chi'_{vd}(H)+$ $+\chi'_{vd}(G)+1, \chi'_{vd}(H)+\chi'_{vd}(G)+2, \dots, \chi'_{vd}(H)+\chi'_{vd}(G)+(m-1)\cdot$ $\cdot\chi'(H)$ in such a way that all the edges incident to the same vertex of matchings are colored differently.

Finally, let us consider the pairs of subgraphs H_i and H_j in $G \oplus H$, which are not neighbors. Note that $W_{i,j}$ is a bipartite graph whose maximum degree is $\Delta(H)$, so by Konig's famous theorem, there exists a proper edge coloring of $W_{i,j}$ which uses exactly $\Delta(H)$ colors. Since H_i and H_j are not neighboring graphs, we have $v_i v_j \notin E(G)$, or equivalently, $v_i v_j \notin E(\bar{G})$. Let $f_{\bar{G}}$ be a proper edge coloring of graph $\bar{G}$ with colors $1,2,\dots,\chi'(\bar{G})$. Denote $M = \chi'_{vd}(H)+\chi'_{vd}(G)+(m-1)\cdot\chi'(H)$. For the bipartite graph $W_{i,j}$, we use a proper edge coloring with colors $M+(f_{\bar{G}}(e)-1)\cdot$ $\Delta(H)+1, M+(f_{\bar{G}}(e)-1)\cdot\Delta(H)+2, \dots, M+f_{\bar{G}}(e)\cdot\Delta(H)+1$.

Since $f_{\bar{G}}$ is a proper edge coloring, we have the coloring of the subgraph $W_{i,j}$ is also a proper edge coloring.

Note that the described coloring uses the colors 1, 2,…, $\chi'_{vd}(G)+$ $+\chi'_{vd}(H)+(m-1)\chi'(G)+\chi'(\bar{G})\cdot\Delta(H)$. Let us denote the described coloring by $f_{G\oplus H}$ and show that for any different vertices $w, z \in V(G \oplus H)$, we have

$$S(w, f_{G\oplus H}) \neq S(z, f_{G\oplus H}).$$

Consider the following two cases:

Case 1. $w, z \in H_i (1 \leq i \leq n)$

According to the definition of the coloring $f_{G\oplus H}$, we have

$$S(w, f_{G\oplus H}) \cap \{1,2,\dots,\chi'_{vd}(H)\} = S(w, f_{H_i}) \text{ and}$$

$$S(z, f_{G\oplus H}) \cap \{1,2,\dots,\chi'_{vd}(H)\} = S(z, f_{H_i}).$$

Since f_{H_i} is a vertex distinguishing edge coloring, we have $S(w, f_{H_i}) \neq S(z, f_{H_i})$, hence $S(w, f_{G\oplus H}) \neq S(z, f_{G\oplus H})$.

Case 2. $w \in H_i, z \in H_j$ $(1 \leq i < j \leq n)$

By the definition of $f_{G \oplus H}$, we have $S(w, f_{G \oplus H}) \cap \{\chi'_{vd}(H) + +1, \chi'_{vd}(H) + 2, \ldots, \chi'_{vd}(H) + \chi'_{vd}(G)\} = S(v_i, f_G)$ and $S(z, f_{G \oplus H}) \cap \{\chi'_{vd}(H) + 1, \chi'_{vd}(H) + 2, \ldots, \chi'_{vd}(H) + \chi'_{vd}(G)\} = S(v_j, f_G)$.

Since f_G is a vertex distinguishing edge coloring and $v_i \neq v_j \in V(G)$, we have $S(v_i, f_G) \neq S(v_j, f_G)$, therefore $S(w, f_{G \oplus H}) \neq S(z, f_{G \oplus H})$.

Conclusion

In our study, we established an upper bound for vertex distinguishing chromatic index of a cartesian sum of graphs, using the chromatic parameters of the individual graphs.

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ВЕРШИННО-РАЗЛИЧАЮЩИЕ РЕБЕРНЫЕ РАСКРАСКИ ДЕКАРТОВЫХ СУММ ГРАФОВ

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АННОТАЦИЯ

Для графа G функция $f\colon E(G) \to N$ называется рёберной раскраской графа G. Рёберная раскраска f графа G называется правильной, если для любых смежных рёбер $e, e' \in E(G), f(e) \neq f(e')$. Если f – правильная раскраска графа G и $v \in V(G)$, то обозначим через $S(v, f)$ множество цветов рёбер, инцидентных вершин v. Правильная раскраска f графа G называется вершинно-различающей, если для любых различных вершин $u, v \in V(G), S(u, f) \neq S(v, f)$. Наименьшее число цветов, необходимое для вершинно-различающей рёберной раскраски графа G, называется вершинно-различающим хроматическим индексом и обозначается $\chi'_{vd}(G)$. В данной работе найдена верхняя оценка вершинно-различающего хроматического индекса декартовых сумм графов.

Ключевые слова: реберная раскраска, сильная реберная раскраска, сильный хроматический индекс, декартова сумма графов.