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ON STRONG CHROMATIC INDEX OF LEXICOGRAPHIC PRODUCTS OF GRAPHS

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ABSTRACT

A strong edge-coloring of a graph G is a mapping $\phi: E(G) \to N$ such that the edges at distance 0 or 1 receive distinct colors. The minimum number of colors required for such a coloring is called the strong chromatic index of G and denoted by $\chi_s'(G)$. In this paper, we study the strong chromatic index of lexicographic product $G \cdot H$ of graphs G and H. In particular, we give tight lower and upper bounds on $\chi_s'(G \cdot H)$.

Keywords: edge-coloring, strong edge-coloring, strong chromatic index, lexicographic product.

Introduction

In this paper, we consider only simple and finite graphs. We use West's book [1] for terminologies and notations not defined here. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in G$ is denoted by $d(v)$ and the maximum degree between the vertices in G by $\Delta(G)$. The chromatic number of a graph G is denoted by $\chi(G)$. We use standard notations P_n, C_n, K_n and $K_{n,m}$ for the path, cycle, complete graph with n vertices, and the complete bipartite graph one part of which has n vertices, and the other one has m vertices, respectively. A *strong edge-coloring* of a graph G is a mapping $\phi: E(G) \to N$ such that the edges at distance 0 or 1 receive distinct colors. The *strong chromatic index* of a graph G is the minimum number of colors required for a strong edge coloring of the graph and is denoted by $\chi_s'(G)$. The concept of strong edge-coloring was introduced by Fouquet and Jolivet in 1983 [2]. In 1985, during a seminar in Prague, Erdős and Nešetřil proposed the following conjecture:

Conjecture 1. *For every graph G with maximum degree $\Delta(G)$,*

$$\chi_s'(G) \le \begin{cases} \frac{5}{4}\Delta(G)^2, \text{if } \Delta(G) \text{ is even,} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1), if\ \Delta(G) \text{ is odd.} \end{cases}$$

The Conjecture 1 was proved to be true for graphs G with $\Delta(G) = 3$ [3, 4], but it is still open for graphs G with $\Delta(G) \ge 4$. In 2006, Cranston [5] showed that $\chi_s'(G) \le 22$ for graphs G with $\Delta(G) = 4$, which was improved to $\chi_s'(G) \le 21$ in 2018 by Huang et al. [6]. In 1990, Chung, Gyárfás, Trotter, and Tuza [7] showed that for graphs G, with significantly large maximum degree $\Delta(G)$, the strong chromatic index is at most $1.998\Delta(G)^2$. The upper bound was improved to $1.93\Delta(G)^2$ [8] in 2018 and later to $1.772\Delta(G)^2$ in 2021 [9].

The lexicographic product of graphs was introduced by Hausdorff in 1914 in the context of ordered sets and topology [10]. The lexicographic product of graphs found different implications in graph theory after and is actively considered in the context of various colorings (See, for example, [11], [12]).

Togni [13] was first to study strong-edge coloring of various graph products. In particular, Togni showed that the following theorems hold true.

Theorem 1 (Togni). Let G and H be two graphs. For the Cartesian product, we have

$$\chi_s'(G \square H) \le \chi_s'(G)\chi(H) + \chi_s'(H)\chi(G).$$

Theorem 2 (Togni). Let G and H be two graphs different from K_2. For the Kronecker product $G \times H$ we have

$$\chi_s'(G \times H) \leq \chi_s'(G)\chi_s'(H).$$

Theorem 3 (Togni). Let G and H be two graphs. For the strong product $G \boxtimes H$ we have

$$\chi_s'(G \boxtimes H) \leq \chi_s'(G)\chi(H) + \chi_s'(H)\chi(G) + 2\chi_s'(G)\chi_s'(H).$$

It this paper we consider strong-edge colorings of lexicographic products of graphs and provide tight lower and upper bounds on strong chromatic index of lexicographic products.

Main Results

Definition 1. *The lexicographic product of graphs G and H is a graph G · H, where V(G · H) = {(v, x) : v ∈ V(G), x ∈ V(H)} and E(G · H) = {((v, x), (u, y)) : (v, u) ∈ E(G) or, v = u and (x, y) ∈ E(H)}.*

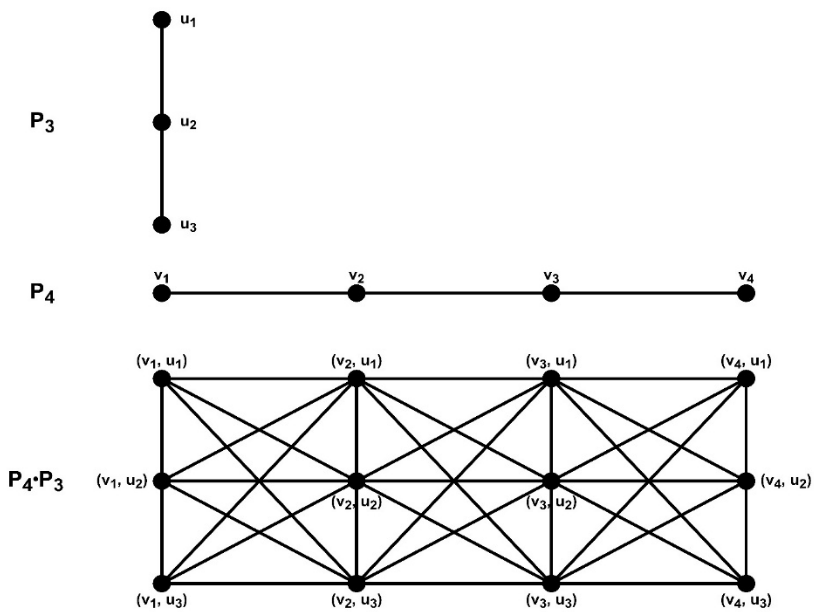

Figure 1. The lexicographical product of P4 and P3.

Figure 2 illustrates the lexicographical product of P4 and P3.

Definition 2. *For graphs H, G, and vertex $v \in V(G)$, denote by vH the subgraph of $G \cdot H$, such that, $V(vH) = \{(v,x) : (v,x) \in V(G \cdot H)\}$ and $E(vH) = \{((v,x),(v,y)) : ((v,x),(v,y)) \in E(G \cdot H)\}$.*

Definition 3. *For graphs H, G, and vertexes $v, u \in V(G)$, denote by $K_{(v,u)}$ the subgraph of $G \cdot H$, such that,*
$V(K_{(v,u)}) = \{(w,x) : w = v \text{ or } w = u, \text{and } (w,x) \in V(G \cdot H)\}$ and
$E(K_{(v,u)}) = \{((v,x),(u,y)) : ((v,x),(u,y)) \in E(G \cdot H)\}$.

It is easy to notice that for the lexicographic product $G \cdot$ H, where G and H are arbitrary graphs, subgraph vH $(v \in V(G))$ is isomorphic to H, and subgraph $K_{(v,u)}(v,u \in V(G))$ is isomorphic to graph $K_{n,n}$, where $n = |V(H)|$.

We begin our considerations with strong edge-colorings of lexicographic products of graphs in special cases.

Lemma 1. *For any graphs $P_n (n \geq 4)$ and H, we have*

$$\chi'_s(P_n \cdot H) = 2\chi'_s(H) + 3|V(H)|^2.$$

Proof. Let $\mathrm{V(P_n)} = \{\mathrm{v_1, v_2, \dots, v_n}\}$ and $\mathrm{E(P_n)} = \{(\mathrm{v_1, v_2}), (\mathrm{v_2, v_3}), \dots, (\mathrm{v_{n-1}, v_n})\}$. For strong edge-coloring of each subgraph $v_i H$ $(1 \leq i \leq n)$ we need at least $\chi'_s(H)$ colors, and $|V(H)|^2$ colors for strong edge-coloring of each subgraph $K_{(v_i,v_j)}$ $((v_i,v_j) \in P_n)$. Consider subgraphs v_2H, v_3H, $K_{(v_1,v_2)}$, $K_{(v_2,v_3)}$, and $K_{(v_3,v_4)}$. Edges from $K_{(v_1,v_2)}$, $K_{(v_2,v_3)}$, and $K_{(v_3,v_4)}$ are at distance 0 or 1 from each other and from edges in v_2H and v_3H. Also, edges from v_2H are at distance 1 from edges in v_3H. Thus, it follows that $\chi'_s(P_n \cdot H) \geq 2\chi'_s(H) + 3|V(H)|^2$.

To complete the proof of the theorem, we construct a strong edge-coloring for $P_n \cdot H$ that uses $2\chi'_s(H) + 3|V(H)|^2$ colors. Let ϕ_P be a strong edge-coloring of P_n with colors $\{1,2,3\}$.

We define edge-coloring ϕ as follows: For each i $(1 \leq i \leq n)$, we color v_iH using colors $\{1,2,\dots,\chi'_s(H)\}$ if i is odd, and using colors

$\{\chi'_s(H)+1, \chi'_s(H)+2, \ldots, 2\chi'_s(H)\}$ if i is even; For each $i(1 \leq i \leq n-1)$, we color $K_{(v_i, v_{i+1})}$ using colors

$$\{2\chi'_s(H) + \left(\phi_P\left((v_i, v_{i+1})\right) - 1\right)|V(H)|^2 + 1, 2\chi'_s(H) + \\ +\left(\phi_P\left((v_i, v_{i+1})\right) - 1\right)|V(H)|^2 + 2, \ldots, 2\chi'_s(H) + \\ +\phi_P\left((v_i, v_{i+1})\right)|V(H)|^2\}.$$

Clearly, ϕ is a strong edge-coloring for graph $P_n \cdot H$ and uses $2\chi'_s(H) + 3|V(H)|^2$ colors. □

Lemma 2. For any graph H, we have

$$\chi'_s(C_5 \cdot H) = 5|V(H)|^2.$$

Proof. Let $V(C_5) = \{v_1, v_2, \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_5\}$ and $E(C_5) = \{(v_1, v_2), (v_2, v_3), (\mathrm{v}_3, \mathrm{v}_4), (\mathrm{v}_4, \mathrm{v}_5), (v_5, v_1)\}$.Consider subgraphs $K_{(v_i, v_j)}$ $\left(\left(v_i, v_j\right) \in E(C_5)\right)$ of graph $C_5 \cdot H$. Since edges from all subgraphs $K_{(v_i, v_j)}$ $\left(\left(v_i, v_j\right) \in E(C_5)\right)$ are at distance 0 or 1 from each other, it follows that $\chi'_s(C_5 \cdot H) \geq 5|V(H)|^2$.

Constructing a strong edge-coloring for graph $C_5 \cdot H$ with $5|V(H)|^2$ colors will complete the proof of the theorem. Let us define an edge-coloring ϕ for graph $C_5 \cdot H$ as follows: color subgraphs $K_{(v_i, v_j)}$ $\left(\left(v_i, v_j\right) \in E(C_5)\right)$ using $5|V(H)|^2$ colors; color subgraph v_1H using colors of subgraph $K_{(v_3, v_4)}$; color subgraph v_2H using colors of subgraph $K_{(v_4, v_5)}$; color subgraph v_3H using colors of subgraph $K_{(v_5, v_1)}$; color subgraph v_4H using colors of subgraph $K_{(v_1, v_2)}$; color subgraph v_5H using colors of subgraph $K_{(v_2, v_3)}$.

It is easy to verify that ϕ is a strong edge-coloring for graph $C_5 \cdot H$ and uses $5|V(H)|^2$ colors.

We continue our considerations with general bounds on strong chromatic index of lexicographic products of graphs.

Theorem 1. *For any graphs G and H, we have*

$$\chi'_s(G)|V(H)|^2 \leq \chi'_s(G \cdot H) \leq \chi(G)\chi'_s(H) + \chi'_s(G)|V(H)|^2.$$

Moreover, the bounds are sharp.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. First of all let us note that subgraphs $K_{(v_i,v_j)}$ and $K_{(v_k,v_l)}$ $((v_i, v_j), (v_k, v_l) \in E(G))$ can be colored with the same colors, only if there exists a strong edge-coloring ϕ_G for graph G, such that $\phi_G\left((v_i, v_j)\right) = \phi_G\left((v_k, v_l)\right)$. Thus, it follows that $\chi'_s(G \cdot H) \geq \chi'_s(G)|V(H)|^2$.

Constructing a strong edge-coloring for graph $G \cdot H$ that uses $\chi(G)\chi'_s(H) + \chi'_s(G)|V(H)|^2$ colors will prove the upper bound. Let f_G be a proper vertex coloring for graph G with colors $\{1,2, \dots, \chi(G)\}$ and ϕ_G be a strong edge-coloring for graph G with colors $\{1,2, \dots, \chi'_s(G)\}$. Let us note that edges from subgraphs v_iH and v_jH $(v_i, v_j \in V(G))$ can be assigned the same colors if $f_G(v_i) = f_G(v_j)$. Also, each subgraph v_iH $(v_i \in V(G))$ can be colored using $\chi'_s(H)$ colors and each subgraph $K_{(v_i,v_j)}$ $((v_i, v_j) \in E(G))$ requires $|V(H)|^2$ colors for strong edge-coloring.

Now we are able to define an edge-coloring ϕ as follows: For each edge $(v_i, v_j) \in E(G)$, we color edges in subgraph $K_{(v_i,v_j)}$ using colors

$$\{(\phi_G((v\_i, v\_j)) - 1)\,|V(H)|^2 + 1, (\phi_G((v\_i, v\_j)) - 1)\,|V(H)|^2 + \\ + 2, \dots, \phi_G((v\_i, v\_j))\,|V(H)|^2\}.$$

For each vertex $v_i \in V(G)$, we color edges in subgraph v_iH using colors

$$\{\chi'_s(G)|V(H)|^2 + (f_G(v_i) - 1)\chi'_s(H) + 1, \chi'_s(G)|V(H)|^2 + \\ +(f_G(v_i) - 1)\chi'_s(H) + 2, \chi'_s(G)|V(H)|^2 + f_G(v_i)\chi'_s(H)\}.$$

Clearly, ϕ is a strong edge-coloring for graph $G \cdot H$ with $\chi(G)\chi'_s(H) + \chi'_s(G)|V(H)|^2\ colors.$

The sharpness of lower and upper bounds follows from Lemma 2 and Lemma 1.

Figure 2 illustrates the strong edge-coloring ϕ of $P_4 \cdot P_3$ described in the proof of Theorem 1.

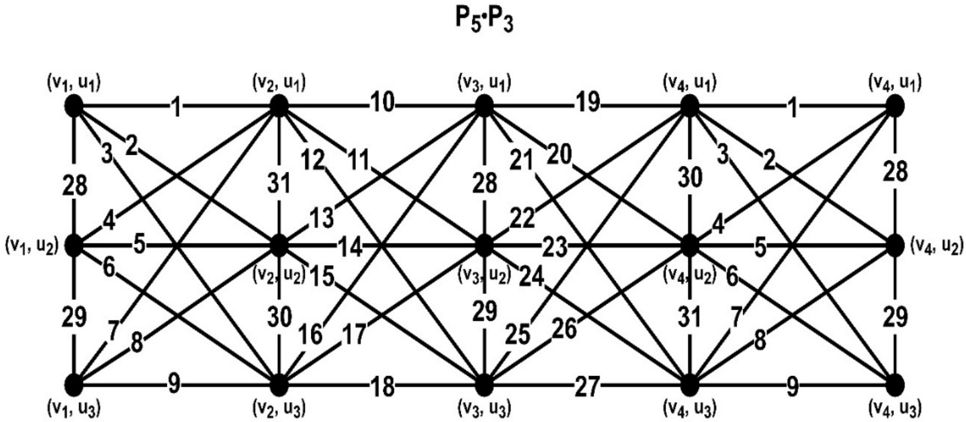

Figure 2. The strong edge-coloring of P5•P3 with 31 colors.

Conclusion

Our study began with an analysis of the strong edge-colorings of the lexicographic products of graphs in special cases. Lemmas 1 and 2 established the exact value of strong chromatic index of lexicographic products of graphs, when first component of the product is a path, and cycle of length 5, respectively. Next, Theorem 1 completed our study of lexicographic products of graphs, by deriving sharp upper and lower bounds in general cases.

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О СИЛЬНОМ ХРОМАТИЧЕСКОМ ИНДЕКСЕ ЛЕКСИКОГРАФИЧЕСКИХ ПРОИЗВЕДЕНИЙ ГРАФОВ

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АННОТАЦИЯ

Сильная реберная раскраска графа G – это отображение $\phi: E(\mathrm{G}) \to N$, такое, что любые два ребра на расстоянии 0 или 1 получают разные цвета. Минимальное количество цветов, необходимое для такого раскрашивания, называется сильным хроматическим индексом графа G и обозначается как $\chi_s'(G)$. В данной работе исследуется сильный хроматический индекс лексикографического произведения графов $G \cdot H$. В частности, мы получаем точные нижние и верхние границы для $\chi_s'(G \cdot H)$.

Ключевые слова: реберная раскраска, сильная реберная раскраска, сильный хроматический индекс, лексикографическое произведение.